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TCP in q-Lorentz Theories

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Abstract. The connection between spin and statistics implied by the continuous Lorentz group together with strong reflection (TCP) is shown to hold also for the q-Lorentz group.

1. Introduction.

The relation of Lorentz invariance to Bose-Einstein and Fermi-Dirac statisticis was discussed by Pauli in several investigations leading to a formulation bsed on the principle of invariance under strong reflection (TCP invariance)¹ as formulated by Schwinger.² There is nothing in these studies, however, to suggest how the commutators of bose fields or the anticommutators of fermion fields would be altered if the Lorentz group were deformed and replaced by another group. This question is now of some interest in connection with speculations about new statistics associated with q-groups. We shall here examine the problem for the q-Lorentz group.

2. The q-Lorentz group and the possibility of q-Commutators.

The two-dimensional representation of the Lorentz group (L) satisfies

$$L^t \epsilon L = L \epsilon L^t = \epsilon \tag{2.1}$$

where L^t is the transposed L and ϵ is the Levi-Civita symbol

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.2}$$

We define the two-dimensional representation of the q-Lorentz group (L_q) by a similar relation:

$$L_q^t \epsilon_q L_q = L_q \epsilon_q L_q^t = \epsilon_q \tag{2.3}$$

where

$$\epsilon_q = \begin{pmatrix} 0 & q_1^{1/2} \\ -q^{1/2} & 0 \end{pmatrix} \qquad q_1 = q^{-1} \tag{2.4}$$

The two groups then agree in the limit q = 1.

To illustrate our question let us consider a vector-spinor field with the conventional Lorentz invariant interaction

$$ar{\psi}A\!\!/\psi$$

and let us pursue conventional theory aside from the introduction of the following time ordered products:

$$T_q(\psi(x)\psi(x')) = \psi(x)\psi(x')$$
 $t > t'$
= $q'\psi(x')\psi(x)$ $t < t'$ vector
= $q''\psi(x')\psi(x)$ $t < t'$ spinor

Then the q-time ordered S-matrix is

$$S^{(q)} = T_q(e^{i \int \mathcal{L}(x)d^4x})$$

where (q) = (q', q''), and the Wick expansion leads to Feynman rules with the following q-propagators³

$$D_{\mu\lambda}^{q'}(x) = \left(g_{\mu\lambda} - \frac{\partial_{\mu}\partial_{\lambda}}{m^{2}}\right) \left(\frac{1}{2\pi}\right)^{4} \left(\frac{1+q'}{2}\right) \int e^{-ikx} \frac{1}{k^{2} - m^{2}} \left[1 + \frac{1-q'}{1+q'} \frac{k_{o}}{\omega}\right] d^{4}k$$

$$S_{\alpha\beta}^{q''}(x) = (\not \partial + m)_{\alpha\beta} \left(\frac{1}{2\pi}\right)^{4} \left(\frac{1-q''}{2}\right) \int e^{-ikx} \frac{1}{k^{2} - m^{2}} \left[1 + \frac{1+q''}{1-q''} \frac{k_{o}}{\omega}\right] d^{4}k .$$

The difference between these two propagators comes from (q', q'') and the sum over polarization and antiparticle states. The final theory may be tested for Lorentz invariance by calculating particle-particle scattering which depends on the photon propagator, and particle-antiparticle annihilation, which depends on the spinor propagator.

In both cases the result is frame dependent, i.e., Lorentz invariance is broken unless q' = 1 for the vector and q'' = -1 for the spinor propagator. This result is then a special case of the Pauli theorem that Lorentz invariance requires commutation and anticommutation rules for integer and half-integer spin respectively.

Our question is now: if the L group is replaced by $L_{q'}$, are q' and q'' still constrained to be +1 and -1 or will one find two new functions: q' = q'(q) and q'' = q''(q)?

3. Strong Reflection.

In the van der Waerden representation of the Lorentz group one utilizes two spinor representations related by complex conjugation, permitting the general tensor to be written as

$$u(n,m) = u_{k_1} \dots u_{k_n} \ v_{\dot{\ell}_1} \dots v_{\dot{\ell}_m}$$
 (3.1)

where the dotted index is the notation for complex conjugation. The tensor (3.1) transforms like the product of n spinors and m complex conjugate spinors. A field characterized by $u_{nm}(x)$ is called fermionic if n + m = odd and bosonic if n + m = even.

The Pauli formulation of strong reflections makes use of the following transformation laws for a general tensor field u(n, m)

$$u'(n,m) = i(-1)^n u(n,m) = -i(-)^m u(n,m)$$
(3.2)

for fermionic fields (n + m = odd) and

$$u'(n,m) = (-)^n u(n,m) = (-)^m u(n,m)$$
(3.3)

for bosonic fields (n + m = even).

The transformations (3.2) and (3.3) preserve the reality condition

$$u(n,m)^* = v(m,n)$$

where * means complex conjugation.

To complete the prescription for strong reflection, one must reflect all coordinates and reverse the order of all fields appearing in a product of several fields. Strong reflection will thus preserve a chronological order of factors.

If the fields appearing in the product are all fermionic, then by applying (3.2) N times we have

$$((u_{n_1m_1}(x_1)\dots u_{n_Nm_N}(x_N))' = i^N(-)^{n_1+\dots+n_N}(u_{n_1m_1}(x_1)\dots u_{n_Nm_N}(x_N))$$

After reversing all coordinates and order of all fields, one has

=
$$fi^N(-)^{n_1+\dots+n_N} (u_{n_N m_N}(-x_N) \dots u_{n_1 m_1}(-x_1))$$

(3.5)

where f is a complex number introduced by reversing the order of the fields in the product and has still to be fixed.

If the individual fields $u_{n_k m_k}(x)$ are fermionic, then the product fields are also either fermionic or bosonic depending on whether N is odd or even, since we may write

$$n = \sum_{1}^{N} n_k \qquad m = \sum_{1}^{N} m_k$$

and

$$n+m = \sum_{1}^{N} (n_k + m_k) = \sum_{1}^{N} (\text{odd})_k = (\text{odd})_k \text{ if } N \text{ is odd}$$
$$= (\text{even}) \text{ if } N \text{ is even }.$$

In (3.5) set

$$u_{nm}=u_{n_1m_1}\ldots u_{n_Nm_N}.$$

Then u_{nm} is either a bosonic or a fermionic field and must satisfy (3.3) or (3.2).

For agreement between (3.5) and (3.2) we require

$$f = i^{1-N} \qquad N \text{ odd} \tag{3.6}$$

and between (3.5) and (3.3)

$$f = i^{-N} \qquad N \text{ even} \tag{3.7}$$

If N = (odd,even) set N = (2p + 1, 2p) then $f = (i^{-2p}, i^{-2p})$ and note

$$(-1)^{\frac{N}{2}(N-1)} = i^{2p(2p-1)} = i^{-2p}$$
 $N \text{ even}$
= $i^{(2p+1)(2p)} = i^{2p} = i^{-2p}$ $N \text{ odd}$ (3.8)

Hence in both cases

$$f = (-1)^{\frac{N}{2}(N-1)} . {(3.9)}$$

If one requires that the order $u_{n_N m_N}(x_N) \dots u_{n_1 m_1}(x_1)$ be obtained from $((u_{n_1 m_1}(x_1) \dots u_{n_N m_N}(x_N))$ as a product of transpositions between neighboring factors each of which introduces the same factor η then

$$f = \eta^{\frac{N}{2}(N-1)} \tag{3.10}$$

as well as (3.9). For N=2 $\eta=-1$. Since η has the same value for all transpositions, $\eta=-1$ for all N.

If there are boson fields in the product there are similar simpler remarks.

In this way strong reflection invariance implies the usual commutation and anticommutation rules between fields.

4. q-Tensors.

The general q-tensor is now defined as a product of spinors that copies the product (3.1), namely:

$$u_{k_1...k_n}^{\dot{\ell}_1...\dot{\ell}_m} = u(k_1)...u(k_n)v(\dot{\ell}_1)...v(\dot{\ell}_n)$$
(4.1)

where u and v are fundamental representations of L_q and \dot{L}_q defined as follows:

$$\epsilon_q = L_q^t \epsilon_q L_q = L_q \epsilon_q L_q^t \tag{4.2}$$

and

$$\dot{\epsilon}_q = \dot{L}_q^t \dot{\epsilon}_q \dot{L}_q = \dot{L}_q \dot{\epsilon}_q \dot{L}_q^t \tag{4.3}$$

where the dot again means complex conjugation and ϵ_q is given by (2.4). We are interested in the case where q is real. Then

$$\dot{\epsilon}_q = \epsilon_q \tag{4.4}$$

$$\epsilon_q = \dot{L}_q^t \epsilon_q \dot{L}_q = \dot{L}_q \epsilon_q \dot{L}_q^t \tag{4.5}$$

One may carry over the usual notation, i.e. (4.1) may be written

$$u_{k_1...k_n}^{\dot{\ell}_1...\dot{\ell}_m} = u(n,m)$$
 (4.6)

where n + m = (even, odd) means (bosonic, fermionic). One also has

$$\dot{u}(n,m) = u(m,n) . \tag{4.7}$$

To describe strong reflections we retain (3.2) and (3.3). These transformations preserve

$$\dot{u}(n,m) = v(m,n) .$$

The argument of the previous section may now be repeated with no changes if u(n, m) is defined by (4.6). Our conclusion is that there is no change in the quantum statistics, as determined by the field commutators and anticommutators, in passing from L to L_q .

5. The q-Light Cone.³

Since earlier discussions of the connection between spin and statistics made use of the distinction between causal and non-causal commutators, it is of interest to see whether the light cone and associated space-like interval are changed in the q-theory.

We summarize a few elements of the q-spinor calculus. The contravariant ϵ metric is

$$\epsilon^{AB}(q) = \epsilon_{AB}(q_1) \ . \tag{5.1}$$

One may define covariant σ matrices

$$(\sigma_q^n)_{B\dot{Y}} = (1, \vec{\sigma})_{B\dot{Y}} \tag{5.2}$$

where $\vec{\sigma}$ abbreviates the usual set of Pauli matrices. Then the matrices contravariant to $(\sigma_q^m)_{B\dot{Y}}$ are

$$(\bar{\sigma}_q^m)^{\dot{X}A} = \epsilon_q^{\dot{X}\dot{Y}} \epsilon_q^{AB} (\sigma_q^m)_{B\dot{Y}} \tag{5.3}$$

and the q-metric (η) is

$$2\eta^{nm} = (\bar{\sigma}_q^m)^{\dot{X}A} (\sigma_q^\eta)_{A\dot{X}} . \tag{5.4}$$

One finds

$$\eta^{mn} = \begin{pmatrix}
\frac{1}{2}(q+q_1) & 0 & 0 & \frac{1}{2}(q-q_1) \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-\frac{1}{2}(q-q_1) & 0 & 0 & -\frac{1}{2}(q+q_1)
\end{pmatrix}$$
(5.5)

The equation of the light cone is then

$$\eta_{mn}X^mX^n = \eta_{00}c^2t^2 + \eta_{33}z^2 - x^2 - y^2 = 0$$

or

$$\frac{1}{2}(q+q_1)c^2t^2 - \frac{1}{2}(q+q_1)z^2 - x^2 - y^2 = 0$$
 (5.6)

or

$$c^q \tau^2 - \zeta^2 - x^2 - y^2 = 0 (5.7)$$

after the rescaling

$$\zeta = \frac{1}{2}(q + q_1)z$$

$$\tau = \frac{1}{2}(q + q_1)t$$
(5.8)

The light cone is then simply rescaled in the q-theory and so the earlier discussion about the connection between spin and statistics should hold, in agreement with the conclusion based on the TCP invariance.

Although L and L_q lead to the same connection between spin and statistics, there are in principle detectable physical differences between the two theories coming from the replacement of the charge conjugation matrix, C, in L by the corresponding matrix (ϵ_q) in L_q , and justified by

$$L^t C L = C$$
$$L^t \epsilon_q L = \epsilon_q$$

in the two-dimensional representation.

The replacement of C by ϵ_q leads to a different relation between particle and antiparticle and in particular it leads to the substitution of q-bilinears for the usual bilinears. These changes, although detectable in principle for weak decays, would probably be masked by radiative corrections.

References.

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